KAN EXTENSION

by

Radheshyam Ota

Ph.D. Student in Mathematics
Studentship Roll No. 50612002
National Institute of Technology
Rourkela - 769 008 (India)

A thesis in conformity with the requirements for

the Degree of Doctor of Philosophy in the

National Institute of Technology, Rourkela.

2009

CERTIFICATE

This is to certify that the thesis entitled "KAN EXTENSION" which is being submitted by *Radheshyam Ota*, Ph.D. Student in Mathematics, Studentship Roll No. 50612002, National Institute of Technology, Rourkela - 769 008 (India), for the award of the Degree of Doctor of Philosophy in Mathematics from National Institute of Technology, is a record of bonafide research work done by him under my supervision. The results embodied in the thesis are new and have not been submitted to any other University or Institution for the award of any Degree or Diploma.

To the best of my knowledge Mr. Ota bears a good moral character and is mentally and physically fit to get the degree.

Professor A. Behera

Supervisor

ACKNOWLEDGEMENTS

I wish to acknowledge my sincere gratitude to my supervisor Dr. Akrur Behera, Professor of Mathematics, National Institute of Technology, Rourkela, for his inspiring guidance and assistance in the preparation of this thesis.

I take the opportunity to express my thankfulness and gratitude to Dr. Sribatsa Nanda, Ex-Professor of Mathematics, Regional Engineering College, Rourkela, for reading the entire thesis and giving some valuable suggestions as and when required.

I thank the Director, National Institute of Technology, Rourkela, for permitting me to avail the necessary facilities of the Institution for the completion of this work.

I feel pleased and privileged to fulfill my parents' ambition, and I am greatly indebted to them for bearing the inconvenience during my thesis work.

Finally, I thank my wife Sujata Mishra, daughter Subhasmita Ota and son Pratyus Kumar Ota for their sacrifice that encouraged me to proceed with the work.

Radheshyam Ota

Ph.D. Student in Mathematics
Studentship Roll No. 50612002
National Institute of Technology
Rourkela - 769 008 (India)

ABSTRACT

Under a set of conditions, the Kan extension of an additive homology theory over smaller admissible subcategory is again a homology theory. A homology theory over the category of simply connected based topological spaces and continuous maps arising through Kan extension process, from an additive homology theory over a smaller subcategory always admits global Adams cocompletion. A cohomology theory on the category of spaces whose homotopy groups are in a Serre class of abelian groups, which is moreover an acyclic ideal of abelian groups, extends to a cohomology theory on the category of 1-connected based topological spaces having the homotopy type of a CW-complex; also under more restrictive conditions a cohomology on the latter category and the Kan extension of its restriction to the former category are suitably related.

TABLE OF CONTENTS

Chapter 0

INTRODUCTION

Kan extensions are universal constructs in Category Theory, a branch of Mathematics. In *Categories for the Working Mathematician*, Saunders Mac Lane titled a section *"All Concepts Are Kan Extensions"*, and went on to write that *"The notion of Kan extensions subsumes all the other fundamental concepts of category theory."*

An early use of (what is now known as) a Kan extension from 1956 was in homological algebra to compute derived functors. They are closely related to adjoints, as well as to limits and ends. They are named after Daniel M. Kan, who constructed certain (Kan) extensions using limits in 1960.

If M is a subset of a nonempty set C, any function $t: M \to A$ to a non-empty set A can be extended to all of C in many ways, but there is no canonical

$$\begin{tikzcd} M \arrow[r, "t"] \arrow[dr] & A \\ & C \arrow[u] \end{tikzcd}$$

or unique way of defining such an extension. However, if $\mathcal{M}$ is a sub category of a category $\mathcal{C}$, each functor $T: \mathcal{M} \to \mathcal{A}$ has in principle two canonical (or extreme)

$$\begin{tikzcd} \mathcal{M} \arrow[r, "T"] \arrow[dr] & \mathcal{A} \\ & \mathcal{C} \arrow[u, shift left, "L"] \arrow[u, shift right, "R"'] \end{tikzcd}$$

"extensions" from $\mathcal{M}$ to functors $L, R: \mathcal{C} \to \mathcal{A}$. These extensions are characterised by the universality of appropriate natural transformations; they need not always exist, but when $\mathcal{M}$ is small and $\mathcal{A}$ is complete and cocomplete they do exist, and can be given as certain limits or as certain ends [26]. These are Kan extensions and are fundamental concepts in Category Theory. With them we find again that each fundamental concept can be expressed in terms of the others [26].

A Kan extension proceeds from the data of three categories $\mathcal{A}, \mathcal{B}, \mathcal{C}$, and two functors $X: \mathcal{A} \to \mathcal{C}$, $F: \mathcal{A} \to \mathcal{B}$, and comes in two varieties: the "left" Kan extension and the "right" Kan extension of X along F.

$$\begin{tikzcd} \mathcal{A} \arrow[r, "X"] \arrow[d, "F"'] & \mathcal{C} \\ \mathcal{B} \arrow[ur] & \end{tikzcd}$$

The *right Kan extension of* X *along* F consists of a functor $R: \mathcal{B} \to \mathcal{C}$ and a natural transformation $\eta: RF \to X$ having the universal property : For any functor $M: \mathcal{B} \to \mathcal{C}$ and natural transformation $\mu: MF \to X$, there exists a unique natural transformation $\delta: M \to R$ making the diagram

$$\begin{tikzcd} MF \arrow[r, "\mu"] \arrow[d, dashed, "\delta_F"'] & X \\ RF \arrow[ur, "\eta"'] & \end{tikzcd}$$

commutative. Here δ_F is the natural transformation with

$$\delta_F(A) = \delta(F(A)): MF(A) \to RF(A)$$

for each object A of $\mathcal{A}$.

The functor R *is often written* $\mathrm{Ran}_F X$.

As with the other universal constructs in category theory, the "left" version of the Kan extension is dual to the "right" one and is obtained by replacing all categories by their opposites. The effect of this on the description above is merely to reverse the direction of the natural transformations. Recall that a *natural transformation* T between the functors $F, G: \mathcal{C} \to \mathcal{D}$ consists of the data of an arrow $T(A): F(A) \to G(A)$ for every object A of $\mathcal{C}$, satisfying a "naturality" property. When we pass to the opposite categories, the source and target of $T(A)$ are swapped, causing T to act in the opposite direction.

The above gives rise to the alternate description: the *left Kan extension of* X *along* F consists of a functor $L: \mathcal{B} \to \mathcal{C}$ and a natural transformation $\epsilon: X \to LF$ having the universal property : For any other functor $M: \mathcal{B} \to \mathcal{C}$ and natural transformation, $\alpha: X \to MF$, there exists a unique natural transformation $\sigma: L \to M$ making the following diagram

$$\begin{array}{ccc} X & \xrightarrow{\epsilon} & LF \\ & \alpha \searrow & \downarrow \sigma_F \\ & & MF \end{array}$$

commutative. Here σ_F is the natural transformation with

$$\sigma_F(A) = \sigma(F(A)) \colon LF(A) \to MF(A)$$

for each object A of $\mathcal{A}$.

The functor L *is often written* $\mathrm{Lan}_F X$.

The use of the word "the" (as in "the left Kan extension") is justified by the fact that, as with all universal constructions, if the object defined exists, then it is unique up to isomorphism. In this case, that means that (for left Kan extensions) if L, M are two left Kan extensions of X along F, and ϵ, α are the corresponding transformations, then there exists a unique *isomorphism* of functors $\sigma \colon L \to M$ such that the diagram, above, commutes. Likewise for right Kan extensions.

Kan extensions as limits (*colimits*)

Suppose that $X \colon \mathcal{A} \to \mathcal{C}$ and $F \colon \mathcal{A} \to \mathcal{B}$ are two functors. If $\mathcal{A}$ is small and $\mathcal{C}$ is cocomplete, then there exists a left Kan extension $\mathrm{Lan}_F X$ of X along F, defined at each object B of $\mathcal{B}$ by

$$\mathrm{Lan}_F X(B) = \underset{f \colon F(A) \to B}{\underset{\to}{\mathrm{colim}}} X(A)$$

where the colimit is taken over the comma category $(F \downarrow B)$. Dually, if $\mathcal{A}$ is small and $\mathcal{C}$ is complete, then right Kan extensions along F exist, and can be computed as limits [26].

Limits (colimits) as Kan extensions

The limit of a functor $F: \mathcal{C} \rightarrow \mathcal{D}$ can be expressed as a Kan extension by $\lim F = \mathrm{Ran}_E F$ where E is the unique functor from $\mathcal{C}$ to $[0]$ (the category with one object and one arrow). The colimit of F can be expressed similarly by $\mathrm{colim} F = \mathrm{Lan}_E F$.

The ubiquity of Kan extensions has developed gradually. The notion of Kan extension is closely associated with adjoint functors. The formal criteria for adjoint functors are due to Benbou in 1965 [4]. The construction of Kan extensions by limits and colimits, in the critical case when the receiving category $\mathcal{A}$ is $\mathcal{Set}$, was achieved by Kan in 1958 [24]. The impact of this construction was understood only gradually. In 1963 Lawvere [25] used these extensions in functorial semantics. Ulmer [36] emphasized their importance, and in an unpublished paper gave the coend formula (without the name coend) for $\mathrm{Lan}_F X$. In 1969, Day and Kelley [6] described Kan extensions in relative categories including abelian categories. In 1970, this idea is further developed by Dubuc [15]; here the coend formula for Kan extensions plays a central role [26]. In this thesis we do not include the coend formula in the topic of investigation.

The concept of Kan extension is an interesting tool for studying many universal objects of Algebraic Topology. In this thesis, the following aspects, untouched so far, of Algebraic Topology, are studied and explored.

In Chapter 1, we recall some definitions and known results on the various aspects of Category Theory and Algebraic Topology, such as, category of fractions, calculus of left (right) fractions, modulo a Serre class $\mathcal{C}$ of abelian groups, etc.,. Some more definitions and results are included in the relevant chapters.

In Chapter 2, we show that under a set of conditions, the Kan extension of an additive homology theory over smaller admissible subcategory is again a homology theory.

In Chapter 3, we show that a homology theory over the category of simply connected based topological spaces and continuous maps arising through Kan extension process from an additive homology theory over a smaller subcategory always admits global Adams cocompletion. We do it in the context of Serre class of abelian groups.

In Chapter 4, using a Serre class of abelian groups, the nature of the Kan extension of a cohomology theory over an admissible category of based topological spaces and maps to a larger admissible category is presented.

In Chapter 5, we present and recall some examples of Kan extension.

Chapter 1

PRE-REQUISITES

In this chapter we recall some definitions and known results on Kan extension. Some more definitions and results are included in the relevant chapters. This chapter serves as the base and background for the study of subsequent chapters and we shall keep on referring back to it as and when required.

1.1 Category of fractions

We recall the abstract definition of the category of fractions [19, 33].

1.1.1 Definition. ([33], p.266) A *Grothendieck universe* (or simply *universe*) is a collection $\mathcal{U}$ of sets such that the following axioms are satisfied:

U(1) : If $\{X_i : i \in I\}$ is a family of sets belonging to $\mathcal{U}$ and I is an element of $\mathcal{U}$, then the union $\bigcup_{i \in I} X_i$ is an element of $\mathcal{U}$.

U(2) : If $x \in \mathcal{U}$, then $\{x\} \in \mathcal{U}$.

U(3) : If $x \in X$ and $X \in \mathcal{U}$, then $x \in \mathcal{U}$.

U(4) : If X is a set belonging to $\mathcal{U}$, then $\mathcal{P}(X)$, the power set of X, is an element of $\mathcal{U}$.

U(5) : If X and Y are elements of $\mathcal{U}$, then $\{X, Y\}$, (X, Y) and $X \times Y$ are elements of $\mathcal{U}$.

We fix a universe $\mathcal{U}$ that contains $\mathbb{N}$ the set of natural numbers (and so $\mathbb{Z}, \mathbb{Q}, \mathbb{R}, \mathbb{C}$).

1.1.2 Definition. ([33], p. 267) A category $\mathcal{C}$ is said to be a *small $\mathcal{U}$-category*, $\mathcal{U}$ being a fixed Grothendieck universe, if the following conditions hold :

S(1). The objects of $\mathcal{C}$ form a set which is an element of $\mathcal{U}$.

S(2). For each pair (X, Y) of objects of $\mathcal{C}$, the set $\mathrm{Hom}_{\mathcal{C}}(X, Y)$ is an element of $\mathcal{U}$.

1.1.3 Definition [33] Let $\mathcal{C}$ be an arbitrary category and S a set of morphisms of $\mathcal{C}$. A *category of fractions* of S is a category denoted by $\mathcal{C}[S^{-1}]$, together with a functor

$$F_S : \mathcal{C} \to \mathcal{C}[S^{-1}]$$

having the following properties:

CF(1): For each $s \in S$, $F_S(s)$ is an isomorphism in $\mathcal{C}[S^{-1}]$.

CF(2): F_S is universal with respect to this property: If $G : \mathcal{C} \to \mathcal{D}$ is a functor such that $G(s)$ is an isomorphism in $\mathcal{D}$ for each $s \in S$, then there is a unique functor $H : \mathcal{C}[S^{-1}] \to \mathcal{D}$ such that $G = H \circ F_S$. Thus we have the following commutative diagram :

$$\begin{tikzcd} \mathcal{C} \arrow[r, "F_S"] \arrow[d, "G"'] & \mathcal{C}[S^{-1}] \arrow[dl, dotted, "H"] \\ \mathcal{D} & \end{tikzcd}$$

1.1.4 Remark. For the explicit construction of the category $\mathcal{C}[S^{-1}]$, we refer to [33]. We content ourselves merely with the observation that the objects of $\mathcal{C}[S^{-1}]$ are same as those of $\mathcal{C}$ and in the case when S admits a calculus of left (right) fractions, the category $\mathcal{C}[S^{-1}]$ can be described very nicely [19 , 33].

1.2 Calculus of left (right) fractions

The concepts of calculus of left fractions and right fractions play a fundamental role in constructing the category of fractions $\mathcal{C}[S^{-1}]$.

1.2.1 Definition. ([33], p. 258) A family of morphisms S in a category $\mathcal{C}$ is said to *admit a calculus of left fractions* if

(a) S is closed under finite compositions and contains identities of $\mathcal{C}$,

(b) any diagram

$$\begin{tikzcd} X \arrow[r, "s"] \arrow[d, "f"'] & Y \\ Z & \end{tikzcd}$$

in $\mathcal{C}$ with $s \in S$ can be completed to a diagram

$$\begin{tikzcd} X \arrow[r, "s"] \arrow[d, "f"'] & Y \arrow[d, "g"] \\ Z \arrow[r, "t"'] & W \end{tikzcd}$$

with $t \in S$ and $tf = gs$,

(c) given

$$X \xrightarrow{s} Y \underset{g}{\overset{f}{\rightrightarrows}} Z \overset{t}{\dashrightarrow} W$$

with $s \in S$ and $fs = gs$, there is a morphism $t: Z \to W$ in S such that $tf = tg$.

A simple characterization for a family of morphisms S to admit a calculus of left fractions is the following:

1.2.2 Theorem. ([10], Theorem 1.3) *Let* S *be a closed family of morphisms of* $\mathcal{C}$ *satisfying*

(a) *if* $uv \in S$ *and* $v \in S$, *then* $u \in S$,

(b) every diagram

$$\begin{tikzcd} \bullet \arrow[r, "s"] \arrow[d, "f"'] & \bullet \\ \bullet & \end{tikzcd}$$

in $\mathcal{C}$ *with* $s \in S$ *can be embedded in a weak push-out diagram*

$$\begin{tikzcd} \bullet \arrow[r, "s"] \arrow[d, "f"'] & \bullet \arrow[d, "g"] \\ \bullet \arrow[r, "t"'] & \bullet \end{tikzcd}$$

with $t \in S$.

Then S *admits a calculus of left fractions*.

The notion of a set of morphisms admitting a calculus of right fractions is defined dually.

1.2.3 Definition. ([33], p. 267) A family S of morphisms in a category $\mathcal{C}$ is said to *admit a calculus of right fractions* if

(a) S is closed under finite compositions and contains identities of $\mathcal{C}$,

(b) any diagram

$$\begin{array}{ccc} & & X \\ & & \downarrow f \\ Z & \underset{s}{\rightarrow} & Y \end{array}$$

in $\mathcal{C}$ with $s \in S$ can be completed to a diagram

$$\begin{array}{ccc} W & \overset{t}{\rightarrow} & X \\ g \downarrow & & \downarrow f \\ Z & \underset{s}{\rightarrow} & Y \end{array}$$

with $t \in S$ and $ft = sg$,

(c) given

$$W \overset{t}{\dashrightarrow} X \underset{g}{\overset{f}{\rightrightarrows}} Y \overset{s}{\rightarrow} Z$$

with $s \in S$ and $sf = sg$, there is a morphism $t: W \rightarrow X$ in S such that $ft = gt$.

The analog of Theorem 1.2.2 follows immediately by duality.

1.2.4 Theorem. ([10], Theorem 1.3*) *Let S be a closed family of morphisms of $\mathcal{C}$ satisfying* :

(a) *if* $vu \in S$ *and* $v \in S$, *then* $u \in S$;

(b) *every diagram*

$$\begin{tikzcd} & \bullet \arrow[d, "f"] \\ \bullet \arrow[r, "s"'] & \bullet \end{tikzcd}$$

in $\mathcal{C}$ with $s \in S$, can be embedded in a weak pull-back diagram

$$\begin{tikzcd} \bullet \arrow[r, "t"] \arrow[d, "g"'] & \bullet \arrow[d, "f"] \\ \bullet \arrow[r, "s"'] & \bullet \end{tikzcd}$$

with $s \in S$.

Then S admits a calculus of right fractions.

1.2.5 Remark. There are some set-theoretic difficulties in constructing the category $\mathcal{C}[S^{-1}]$; these difficulties may be overcome by making some mild hypotheses and using Grothendieck universes. Precisely speaking, the main logical difficulty involved in the construction of a category of fractions and its use, arises from the fact that if the category $\mathcal{C}$ belongs to a particular universe, the category $\mathcal{C}[S^{-1}]$ would, in general, belong to a higher universe ([33], Proposition 19.1.2). In most applications, however, it is necessary that we remain within the given initial universe. This logical difficulty can be overcome by making some kind of assumptions which would ensure that the category of fractions remains within the same universe [11, 12]. Also the following theorem (Theorem 1.2.6) shows that if S admits a calculus of left (right) fractions, then the category of fractions $\mathcal{C}[S^{-1}]$ remains within the same universe as to the one to which the category $\mathcal{C}$ belongs.

1.2.6 Theorem. ([29], Proposition, p. 425) *Let* $\mathcal{C}$ *be a small* $\mathcal{U}$*-category and* S *a set of morphisms of* $\mathcal{C}$ *that admits a calculus of left* (*right*) fractions. *Then* $\mathcal{C}[S^{-1}]$ *is a small* $\mathcal{U}$*-category.*

1.3 Modulo a Serre class $\mathcal{C}$ of abelian groups

In order to make the exposition self-contained, we collect some relevant definitions and theorems involving Serrre classes of abelian groups [22, 34].

1.3.1 Definition. [34] A nonempty class $\mathcal{C}$ of abelian groups is called a *Serre class of abelian groups* if and only if whenever the three-term sequence $A \to B \to C$ of abelian groups is exact and $A, C \in \mathcal{C}$, then $B \in \mathcal{C}$.

The following theorem is easy to verify.

1.3.2 Theorem. ([34], p. 504) *A class $\mathcal{C}$ of abelian groups is a Serre class if and only if it has the following properties*:

(a) $\mathcal{C}$ *contains a trivial group*.

(b) *If $A \in \mathcal{C}$ and $A \approx A'$, then $A' \in \mathcal{C}$.*

(c) *If A is a subgroup of B and $B \in \mathcal{C}$, then $A \in \mathcal{C}$.*

(d) *If A is a subgroup of B and $B \in \mathcal{C}$, then $B/A \in \mathcal{C}$.*

(e) *If $0 \to A \to B \to C \to 0$, is a short exact sequence, with $A, C \in \mathcal{C}$, then $B \in \mathcal{C}$.*

1.3.3 Definition. [34] Let $\mathcal{C}$ be a Serrre class of abelian groups and $A, B \in \mathcal{C}$. A homomorphism $f: A \to B$

(a) is a $\mathcal{C}$-*monomorphism* if $\ker f \in \mathcal{C}$,

(b) is a $\mathcal{C}$-*epimorphism* if $\operatorname{coker} f \in \mathcal{C}$,

(c) is a $\mathcal{C}$-*isomorphism* if it is both a $\mathcal{C}$-monomorphism and a $\mathcal{C}$-epimorphism.

1.3.4 Theorem. [34] *Let $A, B \in \mathcal{C}$ and $f: A \to B$ and $g: B \to C$ be two homomorphisms. Then the following statements are true.*

(a) *If gf is $\mathcal{C}$-monic, then so is f.*

(b) *If gf is $\mathcal{C}$-epic, then so is g.*

1.3.5 Theorem. (The mod-$\mathcal{C}$ Five lemma) [34] *Suppose that*

$$\begin{array}{ccccccccc} A_1 & \to & A_2 & \to & A_3 & \to & A_4 & \to & A_5 \\ f_1 \downarrow & & \downarrow f_2 & & \downarrow f_3 & & \downarrow f_4 & & \downarrow f_5 \\ B_1 & \to & B_2 & \to & B_3 & \to & B_4 & \to & B_5 \end{array}$$

is a diagram of abelian groups and homomorphisms in $\mathcal{C}$ in which the rows are exact and each square is commutative. Then the following are true:

(a) *If f_2, f_4 are $\mathcal{C}$-monomorphisms and f_1 is a $\mathcal{C}$-epimorphism, then f_3 is a $\mathcal{C}$-monomorphism.*

(b) *If* f_2, f_4 *are* $\mathcal{C}$*-epimorphisms and* f_5 *is a* $\mathcal{C}$*-monomorphism, then* f_3 *is a* $\mathcal{C}$*-epimorphism*.

1.3.6 Definition. [34] A Serre class $\mathcal{C}$ is called an *ideal of abelian groups* if $A \in \mathcal{C}$ implies $A \otimes B$, $\mathrm{Tor}(A, B) \in \mathcal{C}$ for arbitrary B.

1.3.7 Definition. [34] A space X is said to be $\mathcal{C}$*-acyclic* if its integral homology groups $H_n(X) \in \mathcal{C}$ for all $n > 0$.

1.3.8 Definition. [34] A Serre class $\mathcal{C}$ of abelian groups is said to be an *acyclic class* if any space of type $(\pi, 1)$ with $\pi \in \mathcal{C}$ is $\mathcal{C}$-acyclic.

Chapter 2

ON KAN EXTENSION OF A HOMOLOGY THEORY

Under a set of conditions, we show that the Kan extension of an additive homology theory over smaller admissible subcategory is again a homology theory.

2.1 Homology theory

2.1.1 Definition. [31] Let $\mathcal{T}$ be the category of based topological spaces and base-point preserving maps. A subcategory $\mathcal{J}$ of $\mathcal{T}$ is said to be *admissible* if it is nonempty, full, closed under the formation of mapping cones and contains (based) homotopy types.

It is evident that an admissible subcategory of $\mathcal{T}$ contains singletons and is closed under suspensions. We denote by $\tilde{\mathcal{J}}$ the homotopy category of $\mathcal{J}$.

2.1.1 Definition. [31] A *homology theory* h on an admissible category $\mathcal{J}$ is a sequence of functors

$$h_n : \mathcal{J} \to \mathcal{A}$$

where $\mathcal{A}$ is the category of abelian groups, together with natural transformations

$$\sigma_n : h_n \cong h_{n+1}\Sigma$$

(Σ denoting the suspension) satisfying the homotopy, suspension and exactness axioms :

(i) If $f_0 \simeq f_1$ then $h_n(f_0) = h_n(f_1)$.

(ii) $\sigma_n : h_n \cong h_{n+1}\Sigma$.

(iii) If $f : X \to Y$ is in $\mathcal{J}$ and C_f is the mapping cone of f and $P_f : Y \to C_f$ is the canonical embedding then

$$h_n(X) \xrightarrow{h_n(f)} h_n(Y) \xrightarrow{h_n(P_f)} h_n(C_f)$$

is exact.

Let $\mathcal{J}_0$ and $\mathcal{J}_1$ be two admissible subcategories of $\mathcal{T}$ with $\mathcal{J}_0 \subset \mathcal{J}_1$ and let h be a homology theory defined on $\mathcal{J}_0$. The left Kan extension of h over $\mathcal{J}_1$, say h', is a functor on $\mathcal{J}_1$ having values in the category of abelian groups. In this chapter we give a set of conditions under which the functor h' is a homology functor. It may be recalled that Deleanu and Hilton have considered this question for cohomology theories [7, 8, 20] and Piccinini [31] has considered the same question for homology theories on stable categories. It will be evident from the conditions imposed and the method of proof of the main result that the description given here is largely dualization of the results obtained in [8].

2.2 The extension procedure

We describe the extension procedures in several steps.

2.2.1 Assumptions on $\mathcal{J}_0$ and $\mathcal{J}_1$

We assume that the categories $\mathcal{J}_0$ and $\mathcal{J}_1$ have the following three properties :

(i) $\tilde{\mathcal{J}}_0$ has weak local push-outs relative to $\tilde{\mathcal{J}}_1$: that is, given a commutative diagram in $\tilde{\mathcal{J}}_1$

$$\begin{array}{ccc} Y & \xrightarrow{u_0} & Y_0 \\ u_1 \downarrow & & \downarrow f_0 \\ Y_1 & \xrightarrow[f_1]{} & X \end{array}$$

with Y_0, Y_1 and Y in $\mathcal{J}_0$, there exists a diagram

$$\begin{tikzcd}
Y \arrow[r, "u_0"] \arrow[d, "u_1"'] & Y_0 \arrow[d, "v_0"] \arrow[ddr, "f_0"] & \\
Y_1 \arrow[r, "v_1"'] \arrow[drr, "f_1"'] & Z \arrow[dr, "g"] & \\
 & & X
\end{tikzcd}$$

in $\tilde{\mathcal{J}}_1$ with Z in $\mathcal{J}_0$.

(ii) The suspension map $\Sigma: \tilde{\mathcal{J}}_1 \to \tilde{\mathcal{J}}_1$ is locally left $\tilde{\mathcal{J}}_0$-adjunctable, which means that condition (a) and (b) below are satisfied.

(a) Given $f: Y \to \Sigma X$ in $\tilde{\mathcal{J}}_1$ with Y in $\tilde{\mathcal{J}}_0$, there exists an object Z in $\tilde{\mathcal{J}}_0$ and $g: Z \to X$ in $\tilde{\mathcal{J}}_1$ and $u: Y \to \Sigma Z$ in $\tilde{\mathcal{J}}_0$ such that $f = (\Sigma g)u$

$$\begin{tikzcd}
Y \arrow[r, "f"] \arrow[d, "u"'] & \Sigma X \\
\Sigma Z \arrow[ur, "\Sigma g"'] &
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
X \\
Z \arrow[u, "g"']
\end{tikzcd}$$

(b) Given diagrams

$$\begin{tikzcd}[column sep=small]
 & \Sigma Z_1 \arrow[dr, "\Sigma g_1"] & & & Z_1 \arrow[d, "g_1"] \\
Y \arrow[ur, "u_1"] \arrow[dr, "u_2"'] & & \Sigma X & & X \\
 & \Sigma Z_2 \arrow[ur, "\Sigma g_2"'] & & & Z_2 \arrow[u, "g_2"']
\end{tikzcd}$$

with Z_1, Z_2 and Y in $\mathcal{J}_0$ and $(\Sigma g_1)u_1 = (\Sigma g_2)u_2$ in $\tilde{\mathcal{J}}_1$, there exist $v_1 \colon Z \to Z_1$, $v_2 \colon Z \to Z_2$, and $u \colon Y \to \Sigma Z$ in $\tilde{\mathcal{J}}_0$ with $g_1 v_1 = g_2 v_2, (\Sigma v_1)u = u_1$ and $(\Sigma v_2)u = u_2$. Thus we have the commutative diagrams

$$\begin{tikzcd}
Z_1 \arrow[dr, "g_1"] & \\
Z \arrow[u, "v_1"] \arrow[d, "v_2"'] & X \\
Z_2 \arrow[ur, "g_2"'] &
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
 & \Sigma Z_1 \arrow[dr, "\Sigma g_1"] & \\
Y \arrow[ur, "u_1"] \arrow[r, "u"] \arrow[dr, "u_2"'] & \Sigma Z \arrow[u, "\Sigma v_1"] \arrow[d, "\Sigma v_2"'] & \Sigma X \\
 & \Sigma Z_2 \arrow[ur, "\Sigma g_2"'] &
\end{tikzcd}$$

(iii) $\mathcal{J}_0$ is closed under finite sums i.e., if Y_1 and Y_2 are in $\mathcal{J}_0$, then so is $Y_1 \vee Y_2$.

We now describe the extension procedure.

2.2.2 Left Kan extension. Let X be an object of $\mathcal{J}_1$. Form the category $\tilde{\mathcal{J}}_{01}(X)$ of all $\tilde{\mathcal{J}}_0$-objects over X: An object in this category is a morphism $f: Y \to X$ in $\tilde{\mathcal{J}}_1$ with Y in $\tilde{\mathcal{J}}_0$; a morphism $u: f_1 \to f_2$ in this category is a morphism $u: Y_1 \to Y_2$ in $\tilde{\mathcal{J}}_0$ such that the diagram

$$\begin{array}{ccc} Y_1 & & \\ & \searrow f_1 & \\ u \downarrow & & X \\ & \nearrow f_2 & \\ Y_2 & & \end{array}$$

is commutative in $\tilde{\mathcal{J}}_1$. It is easy to check that $\tilde{\mathcal{J}}_{01}(X)$ is a category. The *left Kan extension* h_n' of the homology functor h_n is defined as follows :

$$h_n'(X) = \underset{\substack{\rightarrow \\ f}}{\operatorname{colim}}\big(h_n(Y), h_n(u)\big).$$

It is obvious that h_n' is an extension of h_n and defines a covariant functor on $\tilde{\mathcal{J}}_1$ with values in the category of abelian groups.

An alternative description of the groups $h_n'(X)$ can now be given.

2.2.3 An alternative description of $\boldsymbol{h_n'(X)}$. Let

$$P_n(X) = \{(\alpha, f) \mid f: Y \to X \text{ in } \tilde{\mathcal{J}}_1 \text{ with } Y \text{ in } \mathcal{J}_0, \alpha \in h_n(Y)\}.$$

In $P_n(X)$ define a relation $\sim$ by the rule : $(\alpha_1, f_1) \sim (\alpha_2, f_2)$ if and only if there is a commutative diagram

$$\begin{tikzcd} Y_1 \arrow[d, "u_1"'] \arrow[dr, "f_1"] & \\ Y \arrow[r, "f"] & X \\ Y_2 \arrow[u, "u_2"] \arrow[ur, "f_2"'] & \end{tikzcd}$$

in $\tilde{\mathcal{J}}_1$ with

$$Y \text{ in } \mathcal{J}_0, \ u_1, u_2 \text{ in } \tilde{\mathcal{J}}_0,$$

$$f u_1 = f_1, \ f u_2 = f_2 \quad \text{and} \quad h_n(u_1)\alpha_1 = h_n(u_2)\alpha_2.$$

2.2.4 Proposition. *Under the assumptions of* 2.2.1(i), (ii) *and* (iii), $\sim$ *is an equivalence relation on* $P_n(X)$.

Proof. Only transitivity is in question. Suppose that

$$(\alpha_1, f_1) \sim (\alpha_2, f_2) \quad \text{and} \quad (\alpha_2, f_2) \sim (\alpha_3, f_3).$$

We then have two commutative diagrams in $\tilde{\mathcal{J}}_1$ as follows :

$$\begin{tikzcd}
Y_1 \arrow[d, "u_1"'] \arrow[dr, "f_1"] & \\
Y \arrow[r, "f"] & X \\
Y_2 \arrow[u, "u_2"] \arrow[ur, "f_2"'] &
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
Y_2 \arrow[d, "v_2"'] \arrow[dr, "f_2"] & \\
Y' \arrow[r, "f'"] & X \\
Y_3 \arrow[u, "v_3"] \arrow[ur, "f_3"'] &
\end{tikzcd}$$

with Y and Y' in $\mathcal{J}_0$. Moreover,

$$h_n(u_1)\alpha_1 = h_n(u_2)\alpha_2 \quad \text{and} \quad h_n(v_2)\alpha_2 = h_n(v_3)\alpha_3.$$

By condition 2.2.1(i), the commutative diagram

$$\begin{tikzcd}
Y_2 \arrow[r, "v_2"] \arrow[d, "u_2"'] & Y' \arrow[d, "f'"] \\
Y \arrow[r, "f"'] & X
\end{tikzcd}$$

can be embedded in a diagram

$$\begin{tikzcd}
Y_2 \arrow[r, "v_2"] \arrow[d, "u_2"'] & Y' \arrow[d, "v'"] \arrow[ddr, "f'"] & \\
Y \arrow[r, "v"'] \arrow[drr, "f"'] & Z \arrow[dr, "w"] & \\
 & & X
\end{tikzcd}$$

in $\tilde{\mathcal{J}}_1$ with Z in $\mathcal{J}_0$. We consider the diagram

$$\begin{array}{ccccc} & Y_1 & & & \\ vu_1 & \downarrow & \searrow f_1 & & \\ & Z & \xrightarrow{w} & & X \\ v'v_3 & \uparrow & \nearrow f_3 & & \\ & Y_3 & & & \end{array}$$

which is clearly commutative. Moreover

$$\begin{aligned} h_n(vu_1)\alpha_1 &= h_n(v)h_n(u_1)\alpha_1 &&= h_n(v)h_n(u_2)\alpha_2 \\ &= h_n(vu_2)\alpha_2 &&= h_n(v'v_2)\alpha_2 \\ &= h_n(v')h_n(v_2)\alpha_2 &&= h_n(v')h_n(v_3)\alpha_3 \\ &= h_n(v'v_3)\alpha_3. \end{aligned}$$

Hence $(\alpha_1, f_1) \sim (\alpha_3, f_3)$. This completes the proof of Proposition 2.2.4. ■

We denote the equivalence class of (α, f) by $[\alpha, f]$ and denote the set of such equivalence classes by $E_n(X)$.

2.2.5 Proposition. $E_n(X)$ *is a commutative group.*

Proof. We define 'addition' on $E_n(X)$ as follows : Given $[\alpha_1, f_1]$ and $[\alpha_2, f_2]$ in $E_n(X)$ with

$$f_1: Y_1 \to X \quad \text{and} \quad f_2: Y_2 \to X,$$

let

$$i_1: Y_1 \to Y_1 \vee Y_2 \quad \text{and} \quad i_2: Y_2 \to Y_1 \vee Y_2$$

denote the usual injections. Define

$$[\alpha_1, f_1] + [\alpha_2, f_2] = [h_n(i_1)\alpha_1 + h_n(i_2)\alpha_2, f_1 \vee f_2]$$

where

$$f_1 \vee f_2: Y_1 \vee Y_2 \to X$$

is defined as usual. This 'addition' is easily checked to be well-defined, associative and commutative. The additive identity of the group is easily seen to be the element $[0, i]$ where $i{:}{*}{\to}\, X$ is the map that takes the singleton $*$ to the base point of X. We note that $[\alpha, f] = [0, i]$ if and only if we have a commutative diagram

$$\begin{array}{ccc} Y & & \\ u \downarrow & \searrow f & \\ Y_0 & \underset{f_0}{\rightarrow} & X \end{array}$$

in $\tilde{\mathcal{J}}_1$ with $Y_0 \in \mathcal{J}_0$ and $h_n(u)\alpha = 0$. The additive inverse of an element $[\alpha, f]$ is easily seen to be $[-\alpha, f]$. $E_n(X)$ is thus a commutative group. ∎

2.2.5 Theorem. $E_n(X) = h'_n(X)$.

Proof. For a given $f: Y \to X$ in $\tilde{\mathcal{J}}_1$ with Y in $\mathcal{J}_0$, define $i_f: h_n(Y) \to E_n(X)$ by the rule $i_f(\alpha) = [\alpha, f]$. $E_n(X)$ together with the maps $\{i_f\}$ has the required universal property; thus, $E_n(X) = h'_n(X)$. ■

2.2.6 Note. We note that if $g: X \to X'$ is in $\tilde{\mathcal{J}}_1$, then $h'_n(g) = g_*: h'_n(X) \to h'_n(X')$ is defined by the rule $g_*[\alpha, f] = [\alpha, gf]$.

2.3 The Suspension Axiom

We now show that h' satisfies the suspension axiom. First we identify the suspension map

$$\sigma_n: h_n(Y) \to h_{n+1}(\Sigma Y)$$

in $\mathcal{J}_0$ as follows :

$$\sigma[\alpha, f] = [\sigma\alpha, \Sigma f].$$

We then define the suspension isomorphism

$$\sigma'_n: h'_n(X') \to h'_{n+1}(\Sigma X)$$

for any X in $\mathcal{J}_1$ by the rule

$$\sigma'[\alpha, f] = [\sigma\alpha, \Sigma f].$$

2.3.1 Proposition. *The map σ' is onto.*

Proof. If we take $[\beta, g] \in h'_{n+1}(\Sigma X)$, then we have a map $g\colon Y \to \Sigma X$ for some Y in $\mathcal{J}_0$ and $\beta \in h_{n+1}(Y)$. By assumption 2.2.1(ii) we can factorize the map g as

$$\begin{array}{ccc} Y & \xrightarrow{g} & \Sigma X \\ k\downarrow & \nearrow \Sigma u & \\ \Sigma Z & & \end{array}$$

$g = (\Sigma u)k\colon Y \xrightarrow{k} \Sigma Z \xrightarrow{\Sigma u} \Sigma X$, with Z in $\mathcal{J}_0$. Then the commutative diagram

$$\begin{array}{ccc} Y & & \\ k\downarrow & \searrow g & \\ \Sigma Z & \xrightarrow{\Sigma u} & \Sigma X \\ 1_{\Sigma Z}\uparrow & \nearrow \Sigma u & \\ \Sigma Z & & \end{array}$$

shows that

$$\begin{aligned} [\beta, g] &= [h_{n+1}(k)\beta, \Sigma u] = [\sigma\sigma^{-1}(h_{n+1}(k)\beta), \Sigma u] \\ &= \sigma' [\sigma^{-1}(h_{n+1}(k)\beta), u] \end{aligned}$$

Thus σ' is onto. ■

2.3.2 Proposition. *The map* σ' *is one-one.*

Proof. Assume that

$$\sigma'[\alpha, f] = 0,\ f\colon Y \to X \text{ and } \alpha \in h_n(Y).$$

We then have a commutative diagram

$$\begin{tikzcd} \Sigma Y \arrow[d, "u_1"'] \arrow[dr, "\Sigma f"] & \\ Y_1 \arrow[r, "g_1"'] & \Sigma X \end{tikzcd}$$

with Y_1 in $\tilde{\mathcal{J}}_0$ and $h_{n+1}(u_1)(\sigma\alpha) = 0$. We can now factorize g_1 as

$$g_1 = (\Sigma g)u_2\colon Y_1 \xrightarrow{u_2} \Sigma Z \xrightarrow{\Sigma g} \Sigma X.$$

$$\begin{tikzcd} Y_1 \arrow[r, "g_1"] \arrow[d, "u_2"'] & \Sigma X & X \\ \Sigma Z \arrow[ur, "\Sigma g"'] & & Z \arrow[u, "g"'] \end{tikzcd}$$

We thus have a diagram

$$
\begin{tikzcd}
\Sigma Y \arrow[d, "u_1"'] \arrow[drr, "\Sigma f"] & & \\
Y_1 \arrow[rr, "g_1"] & & \Sigma X \\
\Sigma Z \arrow[u, leftarrow, "u_2"] \arrow[urr, "\Sigma g"'] & &
\end{tikzcd}
$$

Let $u = u_2 u_1$; then we have a commutative diagram

$$
\begin{tikzcd}
\Sigma Y \arrow[d, "u"'] \arrow[drr, "\Sigma f"] & & \\
\Sigma Z \arrow[rr, "\Sigma g"'] & & \Sigma X
\end{tikzcd}
$$

Moreover,

$$h_{n+1}(u)(\sigma\alpha) = h_{n+1}(u_2)h_{n+1}(u_1)(\sigma\alpha) = 0.$$

Consider the commutative diagram

$$
\begin{tikzcd}
& \Sigma Y \arrow[dr, "\Sigma f"] & \\
\Sigma Y \arrow[ur, "1_{\Sigma Y}"] \arrow[dr, "u"'] & & \Sigma X \\
& \Sigma Z \arrow[ur, "\Sigma g"'] &
\end{tikzcd}
$$

By assumptions 2.2.1(ii)(b), we have commutative diagrams

$$
\begin{tikzcd}
 & \Sigma Y \arrow[dr, "\Sigma f"] & \\
\Sigma Y \arrow[ur, "1_{\Sigma Y}"] \arrow[r, "v"] \arrow[dr, "u"'] & \Sigma T \arrow[u, "\Sigma k"'] \arrow[d, "\Sigma m"] & \Sigma X \\
 & \Sigma Z \arrow[ur, "\Sigma g"'] &
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
Y \arrow[dr, "f"] & \\
T \arrow[u, "k"] \arrow[d, "m"'] & X \\
Z \arrow[ur, "g"'] &
\end{tikzcd}
$$

with T in $\mathcal{J}_0$. Consider the element $h_{n+1}(v)(\sigma\alpha) \in h_{n+1}(\Sigma T)$. Since

$$\sigma: h_n(T) \xrightarrow{\approx} h_{n+1}(\Sigma T)$$

is an isomorphism, it follows that $h_{n+1}(v)(\sigma\alpha) = \sigma\beta$ for some $\beta \in h_n(T)$. Moreover, $1_{\Sigma Y} = (\Sigma k)v$, so that

$$
\begin{aligned}
\sigma\alpha &= h_{n+1}(1_{\Sigma Y})(\sigma\alpha) \\
&= h_{n+1}(\Sigma k)h_{n+1}(v)(\sigma\alpha) \\
&= h_{n+1}(\Sigma k)(\sigma\beta) \\
&= \sigma h_n(k)(\beta),
\end{aligned}
$$

showing that $\alpha = h_n(k)(\beta)$. Consideration of the diagram

$$
\begin{tikzcd}
T \arrow[d, "k"'] \arrow[dr, "fk"] & \\
Y \arrow[r, "f"'] & X
\end{tikzcd}
$$

shows that $[\beta, fk] = [\alpha, f]$. On the other hand we have

$$\begin{aligned} h_{n+1}(\Sigma m)(\sigma\beta) &= h_{n+1}(\Sigma m)h_{n+1}(v)(\sigma\alpha) \\ &= h_{n+1}(u)(\sigma\alpha) \\ &= \sigma\big(h_n(u)(\alpha)\big) \\ &= 0. \end{aligned}$$

Thus, $h_n(m)(\beta) = 0$. Considering the diagram

$$\begin{array}{ccc} T & & \\ m \downarrow & \searrow fk & \\ Z & \underset{g}{\rightarrow} & X \end{array}$$

we arrive at the conclusion that $[\beta, fk] = 0$, so that $[\alpha, f] = 0$. ■

The above results prove the following.

2.3.3 Theorem. h' *satisfies the suspension axiom*. ■

2.4 The exactness axiom

We shall show now, using the suspension axiom (proved above), that the functor h'_n satisfies the exactness axiom.

2.4.1 Proposition. h_n' *satisfies the exactness axiom.*

Proof. We have to prove that h_n' carries every cokernel sequence

$$A \xrightarrow{g} X \xrightarrow{P_g} C_g$$

in $\tilde{\mathcal{J}}_1$ into an exact sequence. Let $[\alpha, f] \in \ker h_n'(P_g)$ with $f\colon Y \to X$. This implies that $[\alpha, P_g f] = 0$, so that we have a map $u\colon Y \to Y_0$ in $\tilde{\mathcal{J}}_0$ and a map $f_0\colon Y_0 \to C_g$ in $\tilde{\mathcal{J}}_1$ such that $h_n(u)(\alpha) = 0$. Consider the diagram

$$\begin{array}{ccccccccccc}
A & \xrightarrow{g} & X & \xrightarrow{P_g} & C_g & \rightarrow & \Sigma A & \xrightarrow{\Sigma g} & \Sigma X & & \\
 & & f\uparrow & & f_0\uparrow & & f_1\uparrow & & \Sigma f\uparrow & & \\
 & & Y & \xrightarrow[u]{} & Y_0 & \xrightarrow[P_u]{} & C_u & \xrightarrow[Q_u]{} & \Sigma Y & \xrightarrow[\Sigma u]{} & \Sigma Y_0
\end{array}$$

where f_1 and Σf are induced maps between the corresponding terms of the Puppe sequences. Since

$$\sigma\colon h_n(Y) \xrightarrow{\approx} h_{n+1}(\Sigma Y)$$

is an isomorphism, it follows that $h_{n+1}(\Sigma u)(\sigma\alpha) = 0$. But the homology functor on the bottom Puppe sequence is exact; so there is an element $\beta \in h_{n+1}(C_u)$ such that $h_{n+1}(Q_u)\beta = \sigma\alpha$. From the third sequence, we therefore deduce that

$$[\beta, (\Sigma g)f_1] = [\sigma\alpha, \Sigma f].$$

If

$$\gamma = [\beta, f_1] \in h'_{n+1}(\Sigma A),$$

then

$$\begin{aligned} h'_{n+1}(\Sigma g)(\gamma) &= h'_{n+1}(\Sigma g)[\beta, f_1] \\ &= [\beta, (\Sigma g) f_1] = [\sigma\alpha, \Sigma f] \\ &= \sigma' [\alpha, f]. \end{aligned}$$

Since we have an isomorphism

$$\sigma' \colon h'_n(A) \overset{\approx}{\to} h'_{n+1}(\Sigma A)$$

we take

$$\delta^{-1} = (\sigma')^{-1}(\gamma),$$

so that

$$\begin{aligned} h'_n(g)(\delta^{-1}) &= h'_n(g)(\sigma')^{-1}(\gamma) \\ &= (\sigma')^{-1} h'_{n+1}(\Sigma g)(\gamma) \\ &= (\sigma')^{-1}[\sigma\alpha, \Sigma f] \\ &= (\sigma')^{-1}\sigma'[\alpha, f] \\ &= [\alpha, f]. \end{aligned}$$

This shows that

$$\ker h'_n(P_g) \subset \operatorname{image} h'_n(g).$$

To show that

$$\operatorname{image} h'_n(g) \subset \ker h'_n(P_g),$$

it is enough to show that for any element $[\alpha, f] \in h'_n(A)$,

$$h'_n\left(P_g\right) h'_n(g)[\alpha, f] = 0,$$

i.e.,

$$\left[\alpha, P_g g f\right] = 0.$$

In the following commutative diagram

$$\begin{tikzcd}
A \arrow[r, "g"] & X \arrow[r, "P_g"] & C_g \\
Y \arrow[u, "f"] \arrow[r, "1_Y"'] & Y \arrow[u, "gf"] \arrow[r, "P_{1_Y}"'] & C_{1_Y} \arrow[u, "k"']
\end{tikzcd}$$

we have

$$[\alpha, P_g g f] = [h_n(P_{1_Y}) h_n(1_Y)\alpha, k].$$

But the bottom Puppe sequence is carried by h_n into an exact sequence so that

$$h_n(P_{1_Y}) h_n(1_Y)\alpha = 0.$$

Thus we have the desired result. ■

Chapter 3

ADAMS COCOMPLETION

ARISING THROUGH KAN EXTENSION PROCESS

In this chapter, we show that a homology theory over the category of simply connected based topological spaces and continuous maps arising through Kan extension process from an additive homology theory over a smaller subcategory always admits global Adams cocompletion. We do it in the context of Serre class of abelian groups.

3.1 Adams cocompletion

The notion of generalized completion (Adams completion) arose from a general categorical completion process suggested by Adams [1, 2]. Originally, this was considered for admissible categories and generalized homology (or cohomology) theories. Subsequently, this notion has been considered in a more general framework by Deleanu, Frei and Hilton [10], where an arbitrary category and an arbitrary set of morphisms of the category are considered; moreover, they have also suggested the dual notion, namely, the cocompletion (Adams cocompletion) of an object in a category.

It is to be noted that the notions of Adams completions and cocompletions arise via the category of fractions. We recall the following.

3.1.1 Definition. [10] Let $\mathcal{C}$ be a category and S a set of morphisms of $\mathcal{C}$. Let $\mathcal{C}[S^{-1}]$ denote the category of fractions of $\mathcal{C}$ with respect to S and

$$F: \mathcal{C} \to \mathcal{C}[S^{-1}]$$

the canonical functor. Let $\mathcal{S}$ denote the category of sets and functions. Then for a given object Y of $\mathcal{C}$,

$$\mathcal{C}[S^{-1}](Y, -): \mathcal{C} \to \mathcal{S}$$

defines a covariant functor. If this functor is representable by an object Y_S of $\mathcal{C}$, that is,

$$\mathcal{C}[S^{-1}](Y, -) \cong \mathcal{C}(Y_S, -),$$

then Y_S is called the (*generalized*) *Adams cocompletion* of Y with respect to the set of morphisms S or simply the S-*cocompletion* of Y. We shall often refer to Y_S simply as the *cocompletion* of Y [10].

3.2 Some properties of the homology theory h'

Let $\mathcal{T}$ be the category of based topological spaces and base point preserving maps. Let $\mathcal{J}$ be an admissible subcategory of $\mathcal{T}$ and $\tilde{\mathcal{J}}$ be the homotopy category of $\mathcal{J}$ (as described in Chapter 2). Let h be a generalized homology (cohomology) theory defined on $\tilde{\mathcal{J}}$. Let S be the set of morphisms of $\tilde{\mathcal{J}}$ which are carried into isomorphisms by h. If every object of $\tilde{\mathcal{J}}$ admits a cocompletion with respect to S, then we say that the homology theory h admits global Adams cocompletion. Deleanu [11] has shown that any additive theory h on the homotopy category of based CW-complexes and based continuous maps admits global Adams completion.

In this chapter, we show that every homology theory on an admissible category arising from an additive homology theory on a smaller admissible category through Kan extension process always admits global Adams completion. More precisely, let $\mathcal{J}_0$ and $\mathcal{J}_1$ be admissible complete categories with $\mathcal{J}_0 \subset \mathcal{J}_1$ and $\tilde{\mathcal{J}}_1$ be small $\mathcal{U}$-category, where $\mathcal{U}$ is a fixed Grothendieck universe. Let h be an additive homology theory on $\tilde{\mathcal{J}}_0$ such that its Kan extension h' over $\tilde{\mathcal{J}}_1$ is also a homology theory; then we show that h' admits global Adams cocompletion. The proof of this result depends mainly on the particularly nice description of the homology group $h'(X)$ as described in Chapter 2 and additivity of the functor h.

We now choose the set of morphisms in the category $\tilde{\mathcal{J}}_1$. It is well known that $\tilde{\mathcal{J}}_1$ is complete. Let S be the set of morphisms of $\tilde{\mathcal{J}}_1$ which are carried into $\mathcal{C}$-isomorphisms in all dimensions by the homology functor h' where $\mathcal{C}$ is a Serre class of abelian groups which is moreover an acyclic ideal of abelian groups.

We prove the following propositions.

3.2.1 Proposition. S *is saturated.*

Proof. This is evident from Proposition 1.1 ([10], p. 63). ■

3.2.2 Proposition. S *admits a calculus of right fractions*.

Proof. Clearly $\boldsymbol{S}$ is closed under composition. We shall verify conditions (a) and (b) of Theorem 1.2.4. For this, it is enough to prove that every diagram of the form

$$\begin{array}{ccc} & & A \\ & & \downarrow \alpha \\ C & \underset{\gamma}{\rightarrow} & B \end{array}$$

in $\tilde{\mathcal{J}}_1$ with $\gamma \in S$, can be imbedded in a weak pull-back diagram

$$\begin{tikzcd} D \arrow[r, "\delta"] \arrow[d, "\beta"'] & A \arrow[d, "\alpha"] \\ C \arrow[r, "\gamma"'] & B \end{tikzcd}$$

with $\delta \in S$. The essential idea, as has been explained in [16], is to factorize these maps in terms of fibrations and then taking the pull-back of these fibrations. Suppose

$$\alpha = [f] \text{ and } \gamma = [s].$$

We replace the maps f and s by fibrations to get the following diagram

$$\begin{tikzcd} & & A \arrow[d, shift left, "r"] \\ & D \arrow[r, "p"] \arrow[d, "q"'] & P_f \arrow[u, shift left, "\bar{r}"] \arrow[d, "f'"] \\ C \arrow[r, shift right, "t"'] & P_s \arrow[l, shift right, "\bar{t}"'] \arrow[r, "s'"'] & B \end{tikzcd}$$

where f' and s' are fibrations; r and t are mod-$\mathcal{C}$ homotopy equivalences; $\bar{r}$ and $\bar{t}$ are mod-$\mathcal{C}$ homotopy inverses of r and t; P_f and P_s are mapping tracks of f and s; D is the usual pull-back of f' and s' and p, q are the respective projections. So we have

$$f = f'r \quad \text{and} \quad s = s't.$$

Let

$$\delta = [\bar{r}p],\ \beta = [\bar{t}q].$$

Thus

$$\begin{aligned} \alpha\delta &= [f][\bar{r}p] && = [f\bar{r}p] \\ &= [f'r\bar{r}p] && = [f'p] \\ &= [s'q] && = [s't\bar{t}q] \\ &= [s\bar{t}q] && = [s][\bar{t}q] \quad = \gamma\beta. \end{aligned}$$

Moreover, if $\alpha\mu = \gamma\lambda$, let

$$u: U \to A, v: U \to C$$

be in the classes $\mu,\ \lambda$ respectively so that

$$fu \simeq sv \text{ or } f'ru \simeq sv.$$

Let

$$F: U \times I \to B$$

be a homotopy with

$$F_0 = f'ru \quad \text{and} \quad F_1 = sv.$$

Consider the following diagram

$$
\begin{tikzcd}
 & P_f \arrow[d, "f'"] \\
U \arrow[ur, "G_1 \nearrow\nearrow ru"] \arrow[r, "F_1"'] & B
\end{tikzcd}
$$

Since f' is a fibration there exists a homotopy

$$G: U \times I \to P_f$$

such that

(a) $f'G_t = F_t$

(b) $G_0 = ru$.

Thus

$$f'G_1 = F_1 = sv = s'tv.$$

Consider the following diagram

$$
\begin{tikzcd}
U \arrow[dr, "k"] \arrow[drr, "G_1"] \arrow[ddr, "tv"'] & & \\
 & D \arrow[r, "p"] \arrow[d, "q"] & P_f \arrow[d, "f'"] \\
 & P_s \arrow[r, "s'"'] & B
\end{tikzcd}
$$

By the pull-back property of D, there exists a map $k: U \to D$ such that

$$pk = G_1 \simeq ru \quad \text{and} \quad qk = tv.$$

Thus if

$$\rho = [k]: U \to D,$$

then

$$\delta\rho = [\bar{r}p][k] = [\bar{r}pk] = [\bar{r}ru] = [u] = \mu$$

and

$$\beta\delta = [\bar{t}q][k] = [\bar{t}qk] = [\bar{t}tv] = [v] = \lambda.$$

It now remains to be shown that $\delta \in S$. We assume that the map $\alpha: A \to B$ is a fibration with fiber F; then F is also the fiber of the map $\beta: D \to C$ and from the commutative diagram

$$\begin{array}{ccc} F & = & F \\ \downarrow & & \downarrow \\ D & \overset{\delta}{\to} & A \\ \beta \downarrow & & \downarrow \alpha \\ C & \underset{\gamma}{\to} & B \end{array}$$

we have the following commutative diagram

$$\begin{array}{ccccccccccccc}
\cdots & \to & \pi_{m+1}(C) & \to & \pi_m(F) & \to & \pi_m(D) & \to & \pi_m(C) & \to & \pi_{m-1}(F) & \to & \cdots \\
 & & \gamma_* \downarrow & & \| & & \delta_* \downarrow & & \gamma_* \downarrow & & \| & & \\
\cdots & \to & \pi_{m+1}(B) & \to & \pi_m(F) & \to & \pi_m(A) & \to & \pi_m(B) & \to & \pi_{m-1}(F) & \to & \cdots
\end{array}$$

Theorem 1.3.5 implies that δ_* is a $\mathcal{C}$-isomorphism for all $m \geq 0$. Thus $\delta \in S$. ■

3.2.3 Proposition. *Let* $\{s_i: X_i \to Y_i, i \in I\}$ *be a subset of* S. *Then*

$$\prod_{i\in I} s_i : \prod_{i\in I} X_i \to \prod_{i\in I} Y_i$$

is an element of S, *where the index set* I *is an element of* $\mathcal{U}$.

Proof. First we prove that the homology theory h' satisfies the wedge axiom. For each $i \in I$, let p_i denote the homotopy class of the projection map $\prod_{j\in I} X_j \to X_i$. Thus we have group homomorphisms

$$h'(p_i): h'(\prod_{j\in I} X_j) \to h'(X_i),$$

$i \in I$. We need to prove that the group homomorphism

$$\{h'(p_i)\}: h'(\prod_{j\in I} X_j) \to \prod_{i\in I} h'(X_i)$$

is a $\mathcal{C}$- isomorphism.

We first show that $\{h'(p_i)\}$ is a $\mathcal{C}$-monomorphism, i.e., $\ker\{h'(p_i)\} \in \mathcal{C}$ (see [34]; p. 505). Let $[\alpha, f]$ be class in $h'(\prod_{j\in I} X_j)$ such that

$$\{h'(p_i)\}([\alpha, f]) = 0, \text{ i.e., } [\alpha, f p_i] = 0$$

for each $i \in I$, where $f: Y \to \prod_{j \in I} X_j$ is in $\tilde{\mathcal{J}}_1$ with $Y \in \mathcal{J}_0$ and $\alpha \in h(Y)$. Hence for each $i \in I$, there exists a space Y_i in $\mathcal{J}_0$ and maps $u_i: Y \to Y_i$ in $\tilde{\mathcal{J}}_0$ and $f_i: Y_i \to X_i$ such that the following diagram

$$\begin{array}{ccc} Y & & \\ u_i \downarrow & \searrow p_i f & \\ Y_i & \underset{f_i}{\to} & X_i \end{array}$$

commutes, i.e., $f_i u_i = p_i f$ and $h(u_i)(\alpha) = 0$. Let $q_i: \prod_{j \in I} Y_j \to Y_i$ denote the homotopy class of the projection map. By the universal property of the product, there exists a map $u: Y \to \prod_{j \in I} Y_j$ making the diagram

$$\begin{array}{ccc} Y & & \\ u \downarrow & \searrow u_i & \\ \prod_{j \in I} Y_j & \underset{q_i}{\to} & Y_i \end{array}$$

commutative, i.e., $q_i u = u_i$. Hence for each $i \in I$,

$$h(q_i)h(u)(\alpha) = h(u_i)(\alpha) = 0.$$

Since h is an additive homology theory, we note that

$$\{h(q_i)\}: h(\prod_{j\in I} Y_j) \to \sum_{i\in I} h(Y_i)$$

is a $\mathcal{C}$-isomorphism and hence $h(u)(\alpha) = 0$. In the following diagram

$$\begin{tikzcd}
\prod_{j\in I} Y_j \arrow[d, dashed, "v"'] \arrow[dr, "q_i"] & \\
& Y_i \arrow[dr, "f_i"] & \\
\prod_{j\in I} X_j \arrow[rr, "p_i"'] & & X_i
\end{tikzcd}$$

by the definition of the product, there exists a map

$$v: \prod_{j\in I} Y_j \to \prod_{j\in I} X_j$$

in $\tilde{\mathcal{J}}_1$ such that $p_i v = f_i q_i$. Thus

$$p_i v u = f_i q_i u = f_i u_i = p_i f$$

i.e., the following diagram

$$\begin{tikzcd}
& Y \arrow[d, "u"] \arrow[dr, "u_i"] & & \\
f \downarrow & \prod_{j\in I} Y_j \arrow[d, "v"] & Y_i \arrow[dr, "f_i"] & \\
& \prod_{j\in I} X_j \arrow[rr, "p_i"'] & & X_i
\end{tikzcd}$$

is commutative and by the universal property of the product we have $vu = f$. Thus we have the following commutative diagram

$$\begin{array}{ccc} Y & & \\ u \downarrow & \searrow f & \\ \prod_{j\in I} Y_j & \underset{v}{\rightarrow} & \prod_{j\in I} X_j \end{array}$$

with $u \in \tilde{\mathcal{J}}_0$ and $h(u)(\alpha) = 0$. Hence $[\alpha, f] = 0$. Thus $\ker\{h'(p_i)\}$ is a trivial group and hence by Theorem 9.6.1 [34], $\ker\{h'(p_i)\} \in \mathcal{C}$.

Next we show that $\{h'(p_i)\}$ is a $\mathcal{C}$-epimorphism, i.e., $\operatorname{coker}\{h'(p_i)\} \in \mathcal{C}$ (Theorem 9.6.1 [34]). Consider an arbitrary element $\{[\alpha_i, f_i]\}_{i\in I}$ in $\prod_{i\in I} h'(X_i)$ where for each $i \in I$, the class $[\alpha_i, f_i]$ is represented by $f_i: Y_i \to X_i$ with $Y_i \in \mathcal{J}_0$ and $\alpha_i \in h(Y_i)$. Since

$$\{h(q_i)\}: h(\prod_{i\in I} Y_i) \to \prod_{i\in I} h(Y_i)$$

is a $\mathcal{C}$-isomorphism, the element $\{\alpha_i\} \in \prod_{i\in I} h(Y_i)$ corresponds to some element $\alpha \in h(\prod_{i\in I} Y_i)$ such that $\{h(q_i)\}(\alpha) = \{\alpha_i\}$. Thus for each $i \in I$, $h(q_i)(\alpha) = \alpha_i$.

In the following diagram

$$
\begin{tikzcd}
\prod_{j\in I} Y_j \arrow[dd, "q_i"'] \arrow[dr, "q_i"] & & \\
& Y_i \arrow[dr, "f_i"] & \\
Y_i \arrow[rr, "f_i"] & & X_i \\
Y_i \arrow[u, "1_{Y_i}"] \arrow[urr, "f_i"'] & &
\end{tikzcd}
$$

the two triangles commute and for each $i \in I$,

$$h(1_{Y_i})(\alpha_i) = \alpha_i = h(q_i)(\alpha).$$

Hence $[\alpha_i, f_i] = [\alpha_i, f_i q_i]$, for each $i \in I$. By the universal property of q_i, in the following diagram

$$
\begin{tikzcd}
\prod_{j\in I} Y_j \arrow[d, dashed, "g"'] \arrow[dr, "f_i q_i"] & \\
\prod_{j\in I} X_j \arrow[r, "p_i"'] & X_i
\end{tikzcd}
$$

there exists a unique map $g\colon \prod_{j\in I} Y_j \to \prod_{j\in I} X_j$ such that $p_i g = f_i q_i$. Consider the class $[\alpha, g] \in h'\left(\prod_{j\in I} X_j\right)$. For each $i \in I$, we have

$$h'(p_i)([\alpha, g]) = [\alpha, p_i g] = [\alpha, f_i q_i] = [\alpha_i, f_i].$$

Hence

$$\{h'(p_i)\}([\alpha, g]) = \{[\alpha_i, f_i]\}_{i \in I}.$$

So

$$\operatorname{im}\{h'(p_i)\} = \prod_{i \in I} h'(X_i)\,;$$

thus $\operatorname{coker}\{h'(p_i)\}$ is a trivial group and it is in $\mathcal{C}$ by Theorem 9.6.1 [34].

Now we prove that h' satisfies the compatibility axiom with products. Consider the commutative diagram

$$\begin{array}{ccc}
h'(\prod_{i \in I} X_i) & \xrightarrow{\{h'(p_i)\}} & \prod_{i \in I} h'(X_i) \\
h'(\prod_{i \in I} s_i) \downarrow & & \downarrow \prod_{i \in I} h'(s_i) \\
h'(\prod_{i \in I} Y_i) & \xrightarrow[\{h'(p_i')\}]{} & \prod_{i \in I} h'(Y_i)
\end{array}$$

Since $h'(s_i)$ is a $\mathcal{C}$-isomorphism, so is $\prod_{i \in I} h'(s_i)$. By the wedge axiom, as proved above, it follows that $\{h'(p_i)\}$ and $\{h'(p_i')\}$ are $\mathcal{C}$-isomorphisms. Thus it follows that $h'(\prod_{i \in I} s_i)$ is a $\mathcal{C}$-isomorphism. This completes the proof of Proposition 3.2.3. ■

3.3 Existence of Adams cocompletion for the homology theory $\boldsymbol{h'}$

We shall use the following theorem for showing the global existence of the Adams cocompletion of the homology theory h'; the result is essentially dual of Theorem 1 in [12].

3.3.1 Theorem. *Let $\mathcal{C}$ be a complete small $\mathcal{U}$-category ($\mathcal{U}$ is a fixed Grothendieck universe) and S a set of morphisms of $\mathcal{C}$ that admits a calculus of right fractions. Suppose that the following compatibility condition with product is satisfied*:

(P) *If each $s_i: X_i \to Y_i, i \in I$, is an element of S where the index set I is an element of $\mathcal{U}$, then*

$$\prod_{i\in I} s_i : \prod_{i\in I} X_i \to \prod_{i\in I} Y_i$$

is an element of S.

Then every object X of $\mathcal{C}$ has an Adams cocompletion X_S with respect to the set of morphisms S.

From Propositions 3.2.1, 3.2.2 and 3.3.3, it follows that all the conditions of Theorem 3.3.1 are satisfied and so we obtain the following theorem.

3.3.2 Theorem. *The homology theory h' admits global Adams cocompletion.* ■

Chapter 4

KAN EXTENSION OF A COHOMOLOGY THEORY

In this chapter, using a Serre class of abelian groups, the nature of the Kan extension of a cohomology theory over an admissible category of based topological spaces and maps to a larger admissible category is presented.

4.1 Kan extension of a cohomology theory

Let $\mathcal{T}$ be the category of based topological spaces and based maps and $\mathcal{T}_0$, $\mathcal{T}_1$ be two admissible subcategories [22, Appendix, p. 82] of $\mathcal{T}$, i.e., $\mathcal{T}_0$ and $\mathcal{T}_1$ are nonempty, full and closed under the formation of mapping cones and moreover, they contain entire (based) homotopy types. Let $\mathcal{T}_0 \subset \mathcal{T}_1$ and $\tilde{\mathcal{T}}_0$, $\tilde{\mathcal{T}}_1$ be the corresponding homotopy categories.

Let h_0 be a given cohomology theory on $\tilde{\mathcal{T}}_0$. For each X in $\tilde{\mathcal{T}}_1$ we consider the category of $\tilde{\mathcal{T}}_0$-objects under X. An object in this category is a morphism $f: X \to Y$ in $\tilde{\mathcal{T}}_1$ where Y is an object of $\tilde{\mathcal{T}}_0$. A morphism $u: f_0 \to f_1$ in this category is a morphism $u: Y_0 \to Y_1$ such that the diagram

$$\begin{tikzcd} & Y_0 \arrow[dd, "u"] \\ X \arrow[ur, "f_0"] \arrow[dr, "f_1"'] & \\ & Y_1 \end{tikzcd}$$

commutes in $\tilde{\mathcal{T}}_1$. The *Kan extension* h_1 of the cohomology theory h_0 is now defined as follows :

$$h_1^n(X) = \lim_{\substack{\rightarrow \\ f}} \big(h_0^n(Y), h_0^n(u)\big).$$

Obviously, h_1^n is an extension of h_0^n and defines a contravariant functor on $\tilde{\mathcal{T}}_1$ with values in the category of abelian groups. We suppose that this extension defines a cohomology theory on $\tilde{\mathcal{T}}_1$. For various conditions under which this happens see ([22], Appendix). In particular Hilton has proved the following result ([22], Theorem 3.22).

4.1.1 Theorem. *If the category* $\mathcal{T}_0$ *has the following properties, namely,*

(a) $\mathcal{T}_0$ *is closed under finite products,*

(b) $\mathcal{T}_0$ *is closed under* $\tilde{\Omega}$ *where* $\tilde{\Omega}$ *is the right adjoint to the suspension functor* S *in* $\mathcal{T}_1$,

(c) $\mathcal{T}_0$ *has weak pull-backs relative to* $\mathcal{T}_1$,

then the Kan extension h_1 *of the cohomology theory* h_0 *defined on* $\mathcal{T}_0$ *is again a cohomology theory on* $\mathcal{T}_1$.

The purpose of this chapter is to give an application of Theorem 4.1.1 and to explore the nature of the cohomology functor h_1. More precisely, it is shown that if $\mathcal{T}_1$ is the category of 1-connected based topological spaces having the homotopy type of a CW-complex and $\mathcal{T}_0$ is the category of spaces whose homotopy groups are in a Serre class $\mathcal{C}$ of abelian groups, which is moreover an acyclic ideal of abelian groups, then the cohomology theory h_0 extends to a cohomology theory h_1 on $\mathcal{T}_1$ through the Kan extension process and this Kan extension is characterized by the property that if $h_0^n(X) \in \mathcal{C}$ for all X in $\mathcal{T}_0$ then the groups $h_1^n(X)$ are also in $\mathcal{C}$. Furthermore, under more restrictive conditions, it is shown that a cohomology theory h on $\mathcal{T}_1$ and the Kan extension of its restriction to $\mathcal{T}_0$ are suitably related.

4.2 Extension of the cohomology theory h_0

Let $\tilde{\mathcal{T}}_1$ be the homotopy category of 1-connected spaces having the homotopy type of a CW-complex and $\tilde{\mathcal{T}}_0$ be the subcategory of all spaces whose homotopy groups are all in the Serre class $\mathcal{C}$ of abelian groups, which is moreover an acyclic ideal of abelian groups. Clearly $\tilde{\mathcal{T}}_1$ is an admissible category. In order to show that $\tilde{\mathcal{T}}_0$ is one such, it is sufficient to show that $\tilde{\mathcal{T}}_0$ is closed under the formation of mapping cones.

4.2.1 Proposition. $\tilde{\mathcal{T}}_0$ *is admissible.*

Proof. If

$$\alpha = [f]: X \to Y$$

is in $\tilde{\mathcal{T}}_0$, then the mapping cone C_f is clearly 1-connected and consideration of the exact homology sequence of

$$X \xrightarrow{f} Y \xrightarrow{j} C_f$$

yields the fact that the (reduced) homology groups of C_f are all in $\mathcal{C}$ ([34], p. 504) and so are the homotopy groups ([34], Theorem 9.6.15, p. 508), thus showing that $\tilde{\mathcal{T}}_0$ is admissible. ■

The following property of $\tilde{\mathcal{T}}_0$ is easily proved.

4.2.2 Proposition. *Let*

$$\begin{array}{ccc} D & \xrightarrow{[q]} & B \\ [p]\downarrow & & \downarrow[v] \\ A & \xrightarrow[{[u]}]{} & C \end{array}$$

be the usual induced fibration diagram (i.e., *the pull-back*) *of a given diagram*

$$\begin{array}{ccc} & & B \\ & & \downarrow[v] \\ A & \xrightarrow[{[u]}]{} & C \end{array}$$

in $\tilde{\mathcal{T}}_0$ *where* u *and* v *are the usual fibrations. Let* $\widetilde{D}$ *be the universal cover of* D. *Then* $\widetilde{D}$ *is also in* $\tilde{\mathcal{T}}_0$

Proof. Consider the diagram

$$
\begin{array}{ccc}
F & = & F \\
\downarrow & & \downarrow \\
D & \overset{[q]}{\rightarrow} & B \\
[p]\downarrow & & \downarrow [v] \\
A & \underset{[u]}{\rightarrow} & C
\end{array}
$$

where F is the fibre of the fibration p as well as of v. Considering the exact homotopy sequence of the fibration q, it follows that $\pi_i(F) \in \mathcal{C}$ for all $i \geq 1$. Similarly, considering the exact homology sequence of the fibration p, we conclude that $\pi_i(D) \in \mathcal{C}$ for all $i \geq 1$. Since $\widetilde{D}$ is the universal cover of D, it follows that $\widetilde{D}$ is 1-connected and that $\pi_i(\widetilde{D}) \in \mathcal{C}$ for all $i \geq 2$. Moreover D has the homotopy type of a CW-complex [27] and so has $\widetilde{D}$ [37]; thus $\widetilde{D}$ is in $\tilde{\mathcal{T}}_0$ and the proposition is proved. ■

We then have the following theorem.

4.2.3 Theorem. *Let* $\tilde{\mathcal{T}}_0$ *and* $\tilde{\mathcal{T}}_1$ *be as above and* h_0 *be a cohomology theory defined on* $\tilde{\mathcal{T}}_0$. *Then its Kan extension* h_1 *is a cohomology theory on* $\tilde{\mathcal{T}}_1$.

Proof. It is enough to prove that the category $\tilde{\mathcal{T}}_0$ has the properties (a), (b), and (c) of Theorem 4.1.1. That $\tilde{\mathcal{T}}_0$ is closed under finite products is obvious, because $\mathcal{C}$

is an acyclic ideal of abelian groups. That the product of two objects in $\tilde{\mathcal{T}}_0$ has the homotopy type of a CW-complex follows easily from the results of Milnor [27].

In order to show that $\tilde{\mathcal{T}}_0$ is closed under $\tilde{\Omega}$, take Y in $\tilde{\mathcal{T}}_0$ and consider the path space PY, i.e., the space of paths in Y based at the base point y_0. Since PY is contractible, it follows that the diagram

$$\begin{array}{ccc} & & PY \\ & & \downarrow [\pi] \\ PY & \underset{[\pi]}{\rightarrow} & Y \end{array}$$

is in $\tilde{\mathcal{T}}_0$ where π is the usual projection (fibration). We note moreover that for any object X in $\tilde{\mathcal{T}}_1$, $\tilde{\Omega}X$ is just the universal covering space of the loop space ΩX ([22], p. 97). So the pullback

$$\begin{array}{ccc} E & \rightarrow & PY \\ \downarrow & & \downarrow [\pi] \\ PY & \underset{[\pi]}{\rightarrow} & Y \end{array}$$

of the above diagram clearly has the homotopy type of ΩX and by Proposition 4.2.2, $\tilde{\Omega}X$ is in $\tilde{\mathcal{T}}_0$.

Finally we show that $\tilde{\mathcal{T}}_0$ has weak pullbacks relative to $\tilde{\mathcal{T}}_1$. Given a diagram

$$\begin{array}{ccc} & & A \\ & & \downarrow \varphi \\ B & \underset{\psi}{\rightarrow} & C \end{array}$$

in $\tilde{\mathcal{T}}_0$, there is always a diagram

$$\begin{array}{ccc} E & \overset{\alpha}{\rightarrow} & A \\ \beta \downarrow & & \downarrow \varphi \\ B & \underset{\psi}{\rightarrow} & C \end{array}$$

in $\tilde{\mathcal{T}}_1$ satisfying the weak pullback condition (see [22]; Proposition 3.2, p. 86) i.e., for any

$$\gamma: U \rightarrow A \text{ and } \delta: U \rightarrow B$$

in $\tilde{\mathcal{T}}_1$ satisfying $\varphi\gamma = \psi\delta$, there is a map $\theta: U \rightarrow E$ in $\tilde{\mathcal{T}}_1$ such that $\beta\theta = \delta$ and $\alpha\theta = \gamma$. Moreover, E is the pullback of a pair of fibrations. If $e: \tilde{E} \rightarrow E$ is the covering map, then

$$\begin{array}{ccc} \tilde{E} & \xrightarrow{\alpha[e]} & A \\ \beta[e] \downarrow & & \downarrow \varphi \\ B & \underset{\psi}{\rightarrow} & C \end{array}$$

is in $\tilde{\mathcal{T}}_0$ by Proposition 4.2.2. It is also clear that this diagram satisfies the weak pullback condition for any

$$\gamma: U \to A \text{ and } \delta: U \to B$$

Satisfying $\varphi\gamma = \psi\delta$, because U is 1-connected. This completes the proof Theorem 4.2.3. ■

We therefore have the following result.

4.2.4 Theorem. *Let $\mathcal{T}_1$ be the category of 1-connected spaces having the homotopy type of a CW-complex and $\mathcal{T}_0$ be the sucategory of spaces whose homotopy groups belong to $\mathcal{C}$. Then the Kan extension h_1 of any cohomology theory h_0 on $\mathcal{T}_0$ is also a cohomology theory on $\mathcal{T}_1$.*

4.3 A property of h_1

In this section, $\mathcal{T}_0$ and $\mathcal{T}_1$ will be same as described in Theorem 4.2.4. We wish to show that if the cohomology theory h_0 defined on $\mathcal{T}_0$ has the property $h_0^n(X) \in \mathcal{C}$ for all X in $\mathcal{T}_0$ then its Kan extension h_1 over $\mathcal{T}_1$ is characterized by the property that for every X in $\mathcal{T}_1$ the groups $h_1^n(X)$ are also in $\mathcal{C}$. We need the following simple lemmas.

4.3.1 Lemma. *If* A *and* B *are in* $\mathcal{C}$ *and* $\varphi: A \to B$ *is a homomorphism, then* $\ker\varphi$ *and* $\operatorname{im}\varphi$ *are both in* $\mathcal{C}$.

4.3.2 Lemma. *Direct product of objects in* $\mathcal{C}$ *are all in* $\mathcal{C}$.

The proofs of Lemma 4.3.1 and 4.3.2 follow from Theorem 9.6.1 of [34] (p. 504).

4.3.3 Lemma. *Let*

$$A_1 \xrightarrow{\theta} A_2 \xrightarrow{\theta} A_3 \xrightarrow{\theta} \cdots \ \lim_{\rightarrow} A_n$$

be a direct system of abelian groups and homomorphisms in $\mathcal{C}$. *Then* $\lim\limits_{\rightarrow} A_n$ *is also in* $\mathcal{C}$.

Proof. Let

$$\varphi: \bigcup A_n \to \bigcup A_n$$

be the mapping defined by

$$\varphi(a_i) = a_i - \theta(a_i).$$

Then $\lim\limits_{\rightarrow} A_n = \operatorname{coker}\varphi$ and the result follows from Lemmas 4.3.1 and 4.3.2. ■

4.3.4 Theorem. *Let* h_1 *be the cohomology theory on* $\mathcal{T}_1$ *obtained from the cohomology theory* h_0 *on* $\mathcal{T}_0$ *by Kan extension process. If* $h_0^n(X) \in \mathcal{C}$ *for every* n, *then* $h_1^n(X) \in \mathcal{C}$ *also for every* n.

Proof. Since $h_1^n(X)$ is the direct limit of the abelian groups in $\mathcal{C}$, it is also in $\mathcal{C}$. ■

4.4 Relation between a cohomology theory and the Kan extension of its restriction.

In this section we show that under certain conditions, a cohomology theory h on $\mathcal{T}_1$ and the Kan extension of its restriction to $\mathcal{T}_0$ are suitably related. This has been achieved by simply employing the techniques of Hilton [21] by putting coefficients into a cohomology theory. More precisely, we prove the following result (Theorem 4.4.2) where $\mathcal{T}_0$ and $\mathcal{T}_1$ are the same as in Theorem 4.2.4. The following definition will be used in the proof of Theorem 4.4.2.

4.4.1 Definition ([21], p. 198) For a finitely generated abelian group G, LG is defined to be a compact 2-connected CW-complex such that, in ordinary (reduced) cohomology with integer coefficients

$$H^4(LG) = G, \;\; H^i(LG) = 0, \;\; i \neq 4$$

and $h(-;G)$ is defined as the *cohomology theory* h with coefficient group G ([21], p. 201) by $H^n(X;G) = h^{n-4}(X \wedge LG)$.

4.4.2 Theorem. *Let* h *be a representable cohomology theory defined on* $\mathcal{T}_1$ *with coefficient groups in* $\mathcal{C}$ *and let* h_0 *be its restriction to the subcategory* $\mathcal{T}_0$*. Let* h_1 *be the Kan extension of* h_0 *to* $\mathcal{T}_1$*. Then for any* X *in* $\mathcal{T}_1$,

$$h_1^n(X) \cong h^n(X) \otimes C$$

with C *in* $\mathcal{C}$.

Proof. Without loss of generality we can write

$$C = \lim_{\rightarrow} A_k$$

where all the $A_k{}'$s are in $\mathcal{C}$. Let M_n represent h^n. Thus

$$h^n(X) = [X, M_n].$$

Following Hilton [21], we put the coefficient group A_k in the cohomology theory h to obtain

$$H^n(X; A_k) = h^{n-4}(X \wedge LA_k).$$

Let $h^n(X; A_k)$ be represented by Q_k^n. We assume Q_k^n to be 1-connected since $\mathcal{T}_1$ consists of 1-connected space only. We then have

$$\pi_i(Q_k^n) = \pi_{i-4}(M_{n-4}; A_k)$$

and a Universal Coefficient Theorem

$$0 \rightarrow \pi_i(M_{n-4}) \otimes A_k) \rightarrow \pi_i(Q_k^n) \rightarrow \mathrm{Tor}(\pi_{i-3}(M_{n-4}; A_k)) \rightarrow 0.$$

Thus we have

$$\pi_i(Q_k^n) \cong \pi_i(M_{n-4}) \otimes A_k)$$

The homotopy groups of M_n's are all in $\mathcal{C}$ (by assumption); each A_k is also in $\mathcal{C}$; thus Q_k^n is in $\mathcal{T}_0$.

There is also a Universal Coefficient Theorem [21] for any 2-torsion-free abelian group G, namely,

$$0 \to h^n(X) \otimes G \to h^n(X;G) \to \mathrm{Tor}(h^{n-1}(X);G)) \to 0.$$

If we take $G = C$, then

$$\begin{aligned} h^n(X;C) &\cong h^n(X) \otimes C \\ &\cong h^n(X) \otimes (\varinjlim A_k) \\ &\cong \varinjlim h^n(X) \otimes A_k \\ &= \varinjlim h^n(X;A_k). \end{aligned}$$

Since for each k, $h^n(-;A_k)$ is representable by an object of $\mathcal{T}_0$, $h^n(-;C)$ is a direct limit of functors each representable by objects in $\mathcal{T}_0$. Moreover $\mathcal{T}_0$ is $\mathcal{T}_1$-adapted ([8], p.150) and therefore, by Lemma 4.8 of [8], $h^n(-;C)$ is the Kan extension of the functor $h^n(-;C)$ restricted to $\mathcal{T}_0$. But again the same Universal Coefficient Theorem, for any Y in $\mathcal{T}_0$,

$$h^n(Y;C) \cong h^n(Y) \otimes C \cong h^n(Y)$$

since h is a cohomology theory with coefficients in $\mathcal{C}$. Thus we have that

$$h^n(-;C)|_{\mathcal{T}_0} = h_0$$

and that

$$h_1(X) \cong h(X; C) \cong h(X) \otimes C.$$

This completes the proof of the theorem. ■

Chapter 5

EXAMPLES

We present and recall some examples where the assumptions made on the categories $\mathcal{J}_0$ and $\mathcal{J}_1$ are valid.

5.1 Example. In the stable categories, the suspension functor is an isomorphism, so that there is no difficulty in proving that the three axioms hold in such categories [18, 20].

5.2 Example. Let $\mathcal{J}_0$ to be the set of all based topological spaces having the homotopy type of finite *CW*-complexes and $\mathcal{J}_1$ to be the set of all spaces having the homotopy type of *CW*-complexes. It is easy to prove that the three axioms are true in this case.

5.3 Example. Let $\mathcal{J}_1$ be the category of 1-connected based topological spaces having the homotopy type of a CW-complex and $\mathcal{J}_0$ be the subcategory of spaces whose homotopy groups are all finitely generated and P-local, where P denotes a fixed set of primes Then any cohomolgy theory h on $\mathcal{J}_0$ extends to a cohomolgy theory h_1 on $\mathcal{J}_1$ through the Kan extension process [29].

5.4 Example. Let $\mathcal{C}$ be a Serre class of abelian groups and $\mathcal{T}_0(\mathcal{C})$ be the subcategory of $\mathcal{T}_S$ (the category of 1-connected spaces) consisting of spaces whose homotopy groups belong to $\mathcal{C}$ in the sense of Serre [34]. Then any cohomology theory h on $\mathcal{T}_0(\mathcal{C})$ extends to a cohomology theory h_1 on $\mathcal{T}_S$ through Kan extension process ([20], Theorem 4.1).

5.5 Note. Example 5.3 is a sort of variant of Example 5.4.

5.6 Example. Let $\mathcal{CW}_{CF}$ and $\mathcal{CW}_C$ be the categories of connected finite based CW-complexes and connected based CW-complexes respectively. Let h be a homology theory on $\mathcal{CW}_{CF}$. Then the Kan extension h_1 of h to $\mathcal{CW}_C$ is also a homology theory; indeed, it is naturally equivalent to a homology theory defined by a spectrum [32].

5.7 Example. Let $\mathcal{T}$ be a triangulated category and let h, k be two homology theories on $\mathcal{T}$. Let X in $\mathcal{T}$ admit the Adams h-completion X_h, and form the triangle

$$X \xrightarrow{e} X_h \xrightarrow{i} C_e \xrightarrow{j} \sum X$$

in $\mathcal{T}$. Let $\mathcal{T}_0$ be the full subcategory of $\mathcal{T}$ whose objects are those Y such that $h(Y) = 0$. It is plain that $\mathcal{T}_0$ is a triangulated subcategory of $\mathcal{T}$. Let k^0 be the restriction of k to $\mathcal{T}_0$. Then the Kan extension k^1 of k^0 is given by

$$k_n^1(X) = k_{n+1}(C_e)$$

([9], Theorem 4.1).

5.8 Example. Let $\mathcal{S}$ be the stable CW-category and let $\mathcal{S}_0$ be the full subcategory consisting of spaces in the Serre class $\mathcal{C}_P$ of P-torsion groups. Let k be a homology theory on $\mathcal{S}$. The Kan extension k^1 to $\mathcal{S}$ of the restriction of k to $\mathcal{S}_0$, for X in $\mathcal{S}$, is given by

$$k_n^1(X) = k_{n+1}(X; \mathbb{Z}_{P^\infty})$$

([9], Theorem 4.4).

REFERENCES

[1] Adams J.F. : *Idempotent Functors in Homotopy Theory* : Manifolds, Conf. Univ. of Tokyo Press, Tokyo (1973).

[2] Adams J.F. : *Stable Homotopy and Generalized Homology* : The Univ. of Chicago Press, Chicago and London (1974).

[3] Behera A. : *Adams Completion for the Kan extension of a Cohomology Theory* : Indian J. Math., 40(3) (1998), 255 - 267.

[4] Bénabou Jean : Critéries de représentabilité des foncteurs : C.R. Acad. Sci. Paris 260 (1965), 752 - 755.

[5] Campbell H.E.A. : *Mod* 2 *DeRahm Forms and the Steenrod Algebra* : Ph.D Thesis, University of Toronto (1981).

[6] Day B.J. and Kelley G.M. : *Enriched functor categories* : Reports of the Midwest Category Seminar, Vol. III, 178 - 191, Vol. 106, Lecture Notes in Mathematics, Berlin-Heidelberg-New York, Springer (1969).

[7] Deleanu A. and Hilton P. : *Some remarks on general cohomolgy theories* : Math. Scand. 22 (1968), 227 - 240.

[8] Deleanu A. and Hilton P. : *On Kan extension of cohomology theories and Serre class of groups* : Fund. Math. 73 (1971), 143 - 165.

[9] Deleanu A. and Hilton P. : *Localization, Homology and a construction of Adams* : Trans. Amer. Math. Soc.,179 (1973), 349 - 362.

[10] Deleanu A., Frei A. and Hilton P. : *Generalized Adams Completions* : Cahiers de Topologie et Geometrie Differentielle, 15(1) (1974), 61 - 82.

[11] Deleanu A. : *Existence of the Adams completions for CW-complexes* : J. Pure and Appl. Alg., 4(1974), 299 - 308.

[12] Deleanu A. : *Existence of the Adams completions for objects of cocomplete categories* : J. Pure and Appl. Alg., 6 (1975), 31 - 39.

[13] Deo Satya : *Algebraic Topology, A Primer* : Hindustan Book (India), (2003).

[14] Deo S. and Shukla R.A. : *On cohomological dimension and the sum theories* : Acta Math. Hung., 43(1-2) (1984), 17 - 24.

[15] Dubuc E.J. and Street R. : *Dinatural transformations* : Reports of the Midwest Category Seminar, Vol. IV, 126 - 138, Vol. 137, Lecture Notes in Mathematics, Berlin-Heidelberg-New York, Springer (1970).

[16] Eckmann B. and P.J. Hilton : *Unions and intersections in homotopy theory* : Comm. Math. Helv., 38 (1964), 293 - 307.

[17] Dugundji J. : *Topology* : Allyn and Bacon, Neton, MA (1966).

[18] Freyd P. : Stable homotopy : *Proceedings, conference on categorical algebra* : La Jolla, Springer-Verlag (1965), 121 - 172.

[19] Gabriel P. and Zisman M. : *Calculus of Fractions and Homotopy Theory* : Springer-Verlag, New York (1967).

[20] Hilton P. : *On the construction of cohomolgy theories* : Rend. di Math. (1 - 2) 1 (1968), 1 - 14.

[21] Hilton P. : *Putting coefficient into cohomology theory* : Proc. Kon. Neder. Acad. Wet. 73 (1970), 196 - 216.

[22] Hilton P. : *Generalized Cohomology Theory and K-Theory* : Cambridge Univ. Press (1971).

[23] Hilton P.J. and Stammbach U. : *A Course in Homological Algebra* : Springer-Verlag, Berlin (1971).

[24] Kan D.M. : *Adjoint Functors* : Trans. Amer. Math. Soc. Dy (1958), 294 - 32.

[25] Lawvere F.W. : Functorial semantics of algebraic theories : Proc. Nat. Acad. Sci. USA, 50 (1963), 869 - 873.

[26] Mac Lane S. : *Categories for the working Mathematicians* : Springer-Verlag, New York (1971).

[27] Milnor J. : *On spaces having the homotopy type of a CW-complex* : Trans. Amer. Math. Soc. 90 (1959), 272 - 280.

[28] Nanda S. : *On Kan extension of cohomology on the category of p-local spaces* : Rend. Di Math. (1)11, Series VI (1978), 15 - 24.

[29] Nanda S. : *A note on the universe of a category of fractions* : Canad. Math. Bull., 23(4) (1980), 425 - 427.

[30] Nanda S. : *Topics in Algebra* : Tata McGraw-Hill, Delhi (1989).

[31] Piccinini R. A. : *Kan extension of homology theories on stable categories* : Rend. Di Mat. 6 Series VI (1973), 1 - 12.

[32] Piccinini R. A. : *CW-complexes, Homology Theory* : Queen's Papers in Pure and Appl. Math., No. 34, Queen's University, Kingston (1973).

[33] Schubert H. : *Categories* : Springer-Verlag , New York (1972).

[34] Spanier E.H. : *Algebraic Topology* : Springer-Verlag, New York (1982) (McGraw-Hill, New York (1966)).

[35] Switzer R.M. : *Algebraic Topology – Homotopy and Homology* : Springer-Verlag, New York (1975).

[36] Ulmer F. : Properties of dense and relative adjoint functors : J. Algebra 8 (1967), 77 - 95.

[37] Whitehead J.H.C. : *Combinatorial Homotopy* I : Bull. Amer. Math. Soc. 55 (1949), 213 - 245.

[38] Whitehead G. W. : *Elements of Homotopy Theory* : Springer-Verlag, New York (1978).

LIST OF RESEARCH PAPERS PUBLISHED/ACCEPTED/COMMUNICATED

1. Akrur Behera and Radheshyam Ota : *On Kan Extension of Homology and Adams Cocompletion* : Bulletin of the Institute of Mathematics, Academia Sinica (New Series), **4**(1) (2009), 47 - 66.

2. Akrur Behera and Radheshyam Ota : *On Kan Extension of a Cohomology Theory* : Submitted to Indian Journal of Mathematics.

Radheshyam Ota

Ph.D. Student in Mathematics
Studentship Roll No. 50612002
National Institute of Technology
Rourkela - 769 008 (India)